7. Клепиков П.Н., Оскорбин Д.Н. Обобщенные базисы Милнора некоторых четырехмерных метрических алгебр Ли // Ломоносовские чтения на Алтае: фундаментальные проблемы науки и образования : сборник научных статей международной конференции. – 2014. – С. 298-302.
8. Оскорбин Д.Н., Родионов Е.Д., Хромова О.П. О вычислении спектра оператора кривизны конформно (полу)плоских римановых метрик // Известия Алтайского государственного университета. – 2013. – № 1/2 (77). – С. 028–031.
9. Оскорбин Д.Н., Родионов Е.Д. О спектре оператора кривизны трехмерных групп Ли с левоинвариантной римановой метрикой // Доклады Академии наук. – 2013. – Т. 450, №3. – С. 271.

UDC 514.765

Some problems in the theory of homogeneous spaces

E.D.Rodionov
Altai State University, Barnaul

Since a homogeneous space (M,ρ) is geodesically complete, there arises the problem on the behavior of geodesic curves on such spaces, their closure, and on their self-intersection. The following theorem is known in this direction.

Theorem *1 (see [1, 2]). Geodesics on homogeneous spaces are merely closed curves or unclosed curves without self-intersections.*

Moreover, the following theorem is proved in the work [2] of M.V.Mechsheryakov.

Theorem 2. *Geodesic curves of a left-invariant metric on a connected and simply connected nilpotent Lie group are not closed.*

The following two problems arise in a natural way:

Problem 1 *(A. Besse). Find homogeneous Riemannian manifolds all of whose geodesics are closed.*

Problem 2. *Describe homogeneous Riemannian manifolds all of whose geodesics are unclosed.*

For the first time, the Besse problem was considered in the class of normal homogeneous spaces, i.e., those spaces $(G/H, \rho)$ whose homogeneous Riemannian metric ρ is obtained from the Ad(G)-invariant inner product of a Lie group G under the projection $\pi : G \to G/H$. The following theorem was proved in [3].

Theorem *3 (see [3]). Let $(G/H, \rho)$ be a simply connected, normal homogeneous Riemannian spaceall of whose geodesics are closed. Then $(G/H, \rho)$ is isometric to a compact symmetric space of rank 1 (CSROS: S^n, CP^k, HP^m, and CaP^2).*

Later on, by using purely topological methods, the following theorem was proved in [3] for arbitrary homogeneous Riemannian manifolds.

Theorem 4 *(see [3]). A simply connected homogeneous Riemannian manifold all of whose geodesicsare closed and have the same length is isometric to a CSROS.*

Simultaneously, a geometric proof of this theorem having no requirement on the lengths of geodesics was given [4, 5].

Theorem 5 *(see [4, 5]). A simply connected Riemannian manifold all of whose geodesics are closed is isometric to a CSROS.*

The main idea of the proof of Theorem 5 is as follows. If the structure of $(G/H, \rho)$ is complicated, then we seek a flat totally geodesic torus T in $M = G/H$ whose irrational winding is unclosed. Then a finite list of manifolds remains, which is examined step-by-step.

The following conjecture is closely related to the Besse conjecture.

Conjecture 1 *(W.Klinkenberg, see [6]). On a simply connected closed manifold, there exist infinitely many geometrically distinct closed geodesics.*

Although there is still no final answer to the W.Klinkenberg conjecture in the general case, the following theorem holds for homogeneous Riemannian spaces.

Theorem 6 *(see [7]). Let M be a compact, simply connected, homogeneous space not diffeomorphic to a CSROS. Then any Riemannian metric on M admits infinitely many geometrically distinct closed geodesics. If on M, there exists a Riemannian metric ρ such that all geodesics emanating from a certain point p return to this point before a certain common period t, then M is diffeomorphic to a CSROS.*

After the appearance of these works, there naturally arose the problem on the closures of geodesic curves on homogeneous Riemannian spaces. In the case of naturally reductive spaces, it was studied in [8].

Theorem 7 *(see [8]). Let G and H be compact and connected Lie groups, G/H be naturally reductive, $\gamma(t)$ is a geodesic of G/H. Then the closure of $\gamma(t)$ either is simply a closed curve or is isometric to a flat torus of dimension not less than 2.*

The following two examples show that both extreme cases are realized in the theorem in a certain sense.

Example 1. *On the normal homogeneous spaces*
$SU(n+1)/SU(n) \approx S^{2n+1}$, $Sp(n+1)/Sp(n) \approx S^{4n+3}$,
$Spin(9)/Spin(7) \approx S^{15}$, $Sp(n+1)/Sp(n) \cdot S^1 \approx CP^{2n+1}$,
$Sp(2)/SU(2)$; $SU(5)/Sp(2) \cdot S1$, $(SU(3) \cdot U(2)/S^1_{(1,1)})/U(2)$,
the closures of geodesic curves either are simply closed curves or flat, but not totally geodesic tori, and, moreover, the latter always exist.

Example 2. *If G/H is a compact symmetric space, then any of its geodesics either is simply a closed curve or an irrational winding of a certain flat totally geodesic torus of the manifold G/H.*

The following problem naturally arises.

Problem 3. *What is the structure of the closure of geodesic curves of an arbitrary homogeneous Riemannian manifold G/H ?*

Geodesically Orbital Spaces

Definition 1. *A geodesic γ of a Riemannian manifold (M,ρ) is said to be homogeneous if it is an orbit of a one-parameter subgroup $g(t)$ of $Isom(M,\rho)$.*

The following theorem is known.

Theorem 8 *(see [9]). Every homogeneous Riemannian manifold has at least one homogeneous geodesic passing through any point given in advance.*

As is conventional, a geodesic γ is said to be maximal if it is not the restriction of any other geodesic.

Definition 2. *A homogeneous manifold $(G/H, \rho)$ is called a geodesically orbital space if all of its maximal geodesics are homogeneous.*

Remark 6.1. *Naturally reductive and, in particular, normal homogeneous spaces are geodesically orbital spaces.*

There naturally arises the problem on the existence of a geodesically orbital space different from a naturally reductive space. The first such example was constructed by A. Kaplan [10]. This example is the six-dimensional nilpotent Lie group with two-dimensional center (one of the generalized Heisenberg groups) equipped with a certain left-invariant metric.

The class of *weakly symmetric* spaces is closely related to the class of geodesically orbital spaces.

Definition 3. *A Riemannian manifold M is said to be a weakly symmetric space if for every pair of points p, q of M, there exists an isometry of M interchanging the points p and q.*

It is clear that any symmetric space is weakly symmetric and naturally reductive. Also, geodesic spheres in symmetric spaces of rank 1 are weakly symmetric. Note that there exist weakly symmetric spaces which are not even naturally reductive. For example, geodesic spheres in the Cayley projective plane CaP^2 are such spaces. J.Berndt, O.Kowalski, and L.Vanhecke obtained the following result in [11].

Theorem 9 *(see [11]). Every weakly symmetric space M is geodesically orbital.*

Many examples of weakly symmetric spaces were constructed by W.Ziller in [12]. The geodesically orbital spaces of dimension ≤ 6 were classified by O.Kowalski and L.Vanhecke in [13]. It turns out that all geodesically orbital spaces of dimension ≤ 5 are naturally reductive. At the same time, in the case of dimension equal to 6, there exist three- and two-parameter families of geodesically orbital spaces that are not naturally reductive. Among these families, there is the compact symmetric homogeneous space SO(5)/U(2) having a two-parameter family of invariant metrics.

The structure of geodesically orbital spaces was also studied by Gordon in [14], where the case of nilpotent Lie groups with left-invariant Riemannian metric was studied in detail.

Among recent works, we can mention the work D.Alekseevsky and A.Arvanitoyeorgos [15] devoted to metrics with homogeneous geodesics on flag manifolds. In particular, the following theorem was proved in [15].

Theorem 10. *Let $M = G/H$ be a Riemannian flag manifold of a classical Lie group G. Assume that M is a geodesically orbital space with respect to a G-invariant Riemannian metric different from the standard metric. Then M is $SO(2l+1)/U(l-m) \cdot SO(2m+1)$ for certain $l \geq 2$ and $m \geq 0$.*

At the same time, the following problem remains unsolved.

Problem 4. *Classify all geodesically orbital spaces.*

In concluding we mention the works about other problems in the theory of homogeneous spaces [16–25].

This work was supported by RFBR (grants: № 16-01-00336A, № 16-31-00048mol_a), the Russian Federation Ministry of Education (Project Code: 1148).

References

1. Lacomba E. Mechanical systems with symmetry on homogeneous spaces // Trans. Amer. Math. Soc., 185, 477–491 (1974).

2. Meshcheryakov M.V. Several remarks on Hamiltonian flows on homogeneous spaces // Usp. Mat. Nauk, 40, No. 3, 215–216 (1985).

3. Besse A.L., Manifolds All of Whose Geodesics are Closed [Russian translation], Mir, Moscow (1981).

4. Rodionov E. D. Homogeneous Riemannian Z-manifold // Sib. Mat. Zh., 22, №2, 191–197 (1981).

5. E. D. Rodionov, Structure of Homogeneous Riemannian Z-Manifolds, Dissertation [in Russian], Inst. Math. Sib. Department Russ. Acad. Sci., Novosibirsk (1982).

6. W. Klingenberg, Lectures on Closed Geodesics [Russan translation], Mir, Moscow (1982).

7. McCleary Y. and Ziller W. On the free loop space of homogeneous spaces // Amer. J. Math., 109, 765–781 (1987); correction: Amer. J. Math., 113, 375–377 (1991).

8. Rodionov E. D. Closures of geodesic curves of compact naturally reductive spaces // Proc. Conf. Dedicated to the Memory of N.I. Lobachevskii [in Russian], Kazan (1993).

9. Kowalski O.and Szenthe J. On the existence of homogeneous geodesics in homogeneous Riemannian manifolds // Geom. Dedic., 81, 209–214 (2000); correction: Geom. Dedic., 84, 331–332 (2001).

10. Kaplan A. On the geometry of groups of Heisenberg type // Bull. London Math. Soc., 15, 35–42 (1983).

11. Berndt J., Kowalski O. and Vanhecke L. Geodesics in weakly symmetric spaces // Ann. Global Anal. Geom., 15, 153–156 (1997).

12. Wang M. and Ziller W. On isotropy irreducible Riemannian manifolds // Acta Math., 166, 223–261 (1991).

13. Kowalski O. and Vanhecke L. Riemannian manifolds with homogeneous geodesics // Boll. Unione Mat. Ital. VII. Ser. B, 5, №1, 189–246 (1991).

14. Gordon C. Homogeneous manifolds whose geodesics are orbits // Topics in Geometry. In Memory of J. D'Atri, Birkh.auser, Boston (1996), pp. 155–174.

15. D.Alekseevsky and A.Arvanitoyeorgos, «Metrics with homogeneous geodesics on flag manifolds», Commun. Math. Univ. Carol., 43, №2, pp. 189–199 (2002).

16. Rodionov E.D. Standard homogeneous Einstein manifolds // Доклады Академии наук. – 1993. – Т. 328, № 2. – С. 147.

17. Родионов Е.Д., Славский В.В. Одномерная секционная кривизна римановых многообразий // Доклады Академии наук. – 2002. – Т. 387, № 4. – С. 454.

18. Родионов Е.Д., Славский В.В., Чибрикова Л.Н. Левоинвариантные лоренцевы метрики на 3-мерных группах Ли с нулевым квадратом длины тензора Схоутена-Вейля // Вестник Алтайского государственного педагогического университета. – 2004. – № 4–3. – С. 53–60.

19. Воронов Д.С., Родионов Е.Д. Левоинвариантные римановы метрики на четырехмерных неунимодулярных группах Ли с нулевой дивергенцией тензора Вейля // Доклады Академии наук. – 2010. – Т. 432, № 3. – С. 301–303.

20. Гладунова О.П., Родионов Е.Д., Славский В.В. О спектре оператора кривизны конформно плоских римановых многообразий // Доклады Академии наук. – 2013. – Т. 450, № 2. – С. 140.

21. Никоноров Ю.Г., Родионов Е.Д., Славский В.В. Геометрия однородных римановых многообразий // Современная математика и ее приложения. – 2006. – Т. 37, С. 1–78.

22. Клепиков П.Н., Оскорбин Д.Н., Родионов Е.Д. Об однородных солитонах Риччи на четырехмерных группах Ли с левоинвариантной римановой метрикой // Доклады Академии наук. – 2015. – Т. 465, № 3. – С. 281.

23. Родионов Е.Д. Однородные римановы многообразия с метрикой Эйнштейна / автореферат диссертации на соискание ученой степени доктора физико-математических наук / Новосибирск, 1994.

24. Rodionov E.D. Simply connected compact five-dimensional homogeneous Einstein manifolds // Siberian Mathematical Journal. – 1994. – Т. 35. – С.163–168.

25. Nikonorov Y.G., Rodionov E.D. Standard homogeneous Einstein manifolds and Diophantine equations // Archiv der Mathematik. – 1996. – Т. 32. – С. 23–26.

УДК 514.172

Двойственность для конформно-плоских метрик неотрицательной кривизны

Е.Д. Родионов[1], В.В. Славский[2], М.В. Куркина[2]
[1] АлтГУ, г. Барнаул; [2]ЮГУ, г. Ханты-Мансийск

В теории выпуклых подмножеств векторного пространства важную роль играет двойственность Минковского [1]. Для конформно-плоских метрик можно определить аналог этого понятия.